Capítulo 6: Análisis en el dominio del tiempo de sistemas de primer y segundo orden

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Análisis en el dominio del tiempo de sistemas de primer y segundo orden

- Las propiedades dinámicas de las plantas pueden ser aproximadas por las características temporales de sistemas más simples.
  - Se entiende por modelos simples, aquellos que definen su dinámica por ecuaciones diferenciales lineales de primer o de segundo orden.
- Desde el punto de vista del análisis, al reducir el modelo se podrá predecir sus características temporales empleando expresiones matemáticas de los modelos sencillos.
- Desde el diseño, se suele emplear las medidas de las características temporales de los modelos simples para fijar los requisitos del comportamiento dinámico de los sistemas a compensar.

Sistemas de primer orden

- La función de transferencia de un sistema de primer orden es:

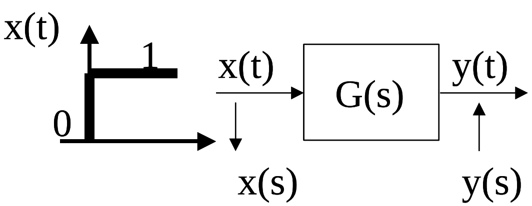
$$G(s) = \frac{N(s)}{(s+a)}$$

- En el caso más simple, el numerador corresponde a una ganancia:

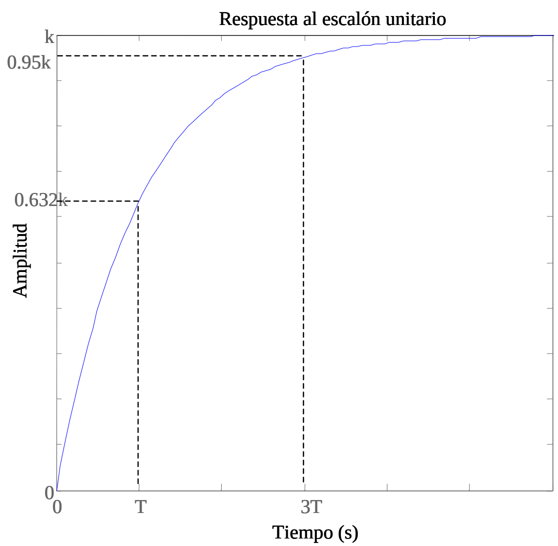
$$G(s) = \frac{Y(s)}{X(s)} = \frac{k}{1+Ts} = \frac{k/T}{s+\frac{1}{T}}$$

Respuesta temporal ante la entrada en escalón

- Analítica & transformadas de Laplace



$$Y(s) = \frac{1}{s} \cdot \frac{k}{1+sT}$$



$$Y(s) = \frac{1}{s} \cdot \frac{k}{1+sT} = k\left(\frac{1}{s} \cdot \frac{1}{1+sT}\right) = \frac{k_1}{s} + \frac{k_2}{s+\frac{1}{T}}$$

$$\boxed{y(t) = k\left(1 - e^{-t/T}\right)}$$

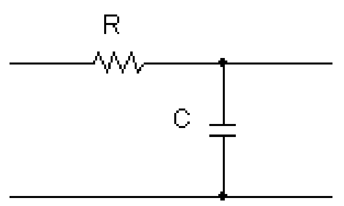
Valor $t = T$: $y(t = T) = k\left(1 - e^{-1}\right) = 0.632\,k$

Valor $t = 3T$: $y(t = 3T) = k\left(1 - e^{-3}\right) = 0.95k$

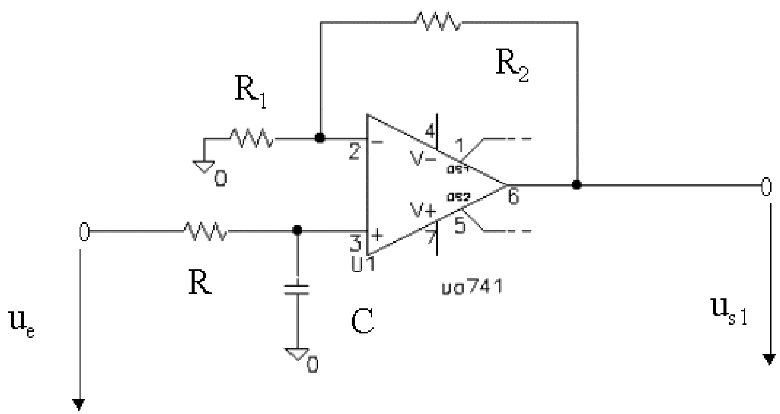
Valor inicial: $\lim_{s\to\infty} sY(s) = 0$ $\lim_{t\to 0} y(t) = 0$

Valor final: $\lim_{s\to 0} sY(s) = k$ $\lim_{t\to\infty} y(t) = k$

Ejemplos

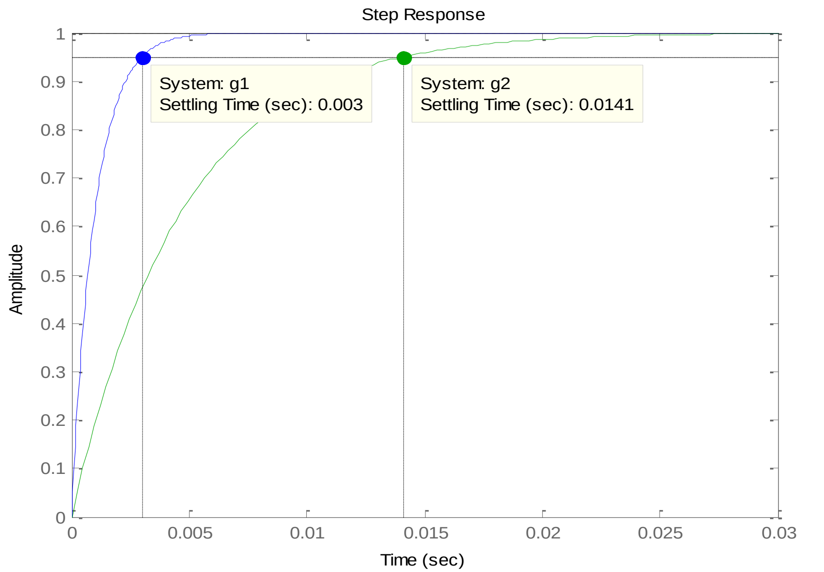


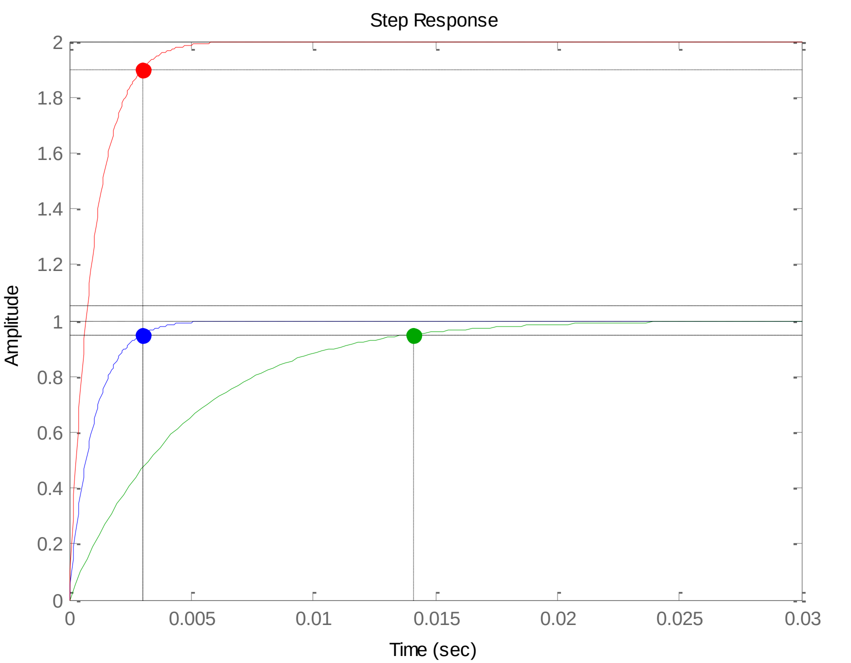
R=100k ó 470k C=10nF



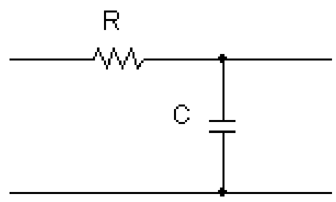
R=100k C=10nF

R_1=33k, R_2=33k

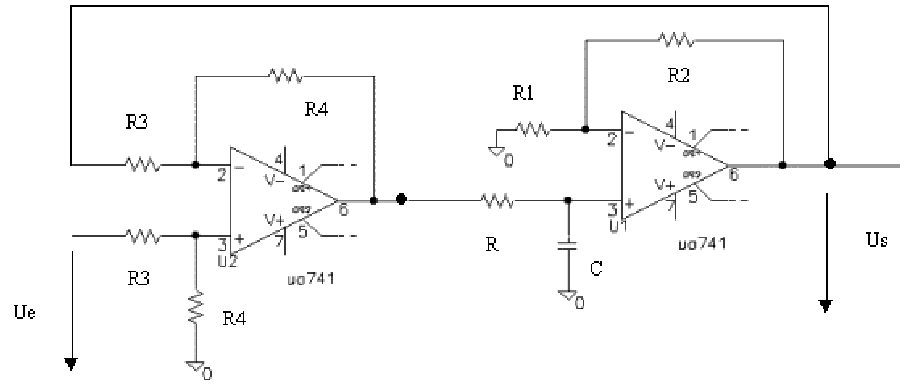


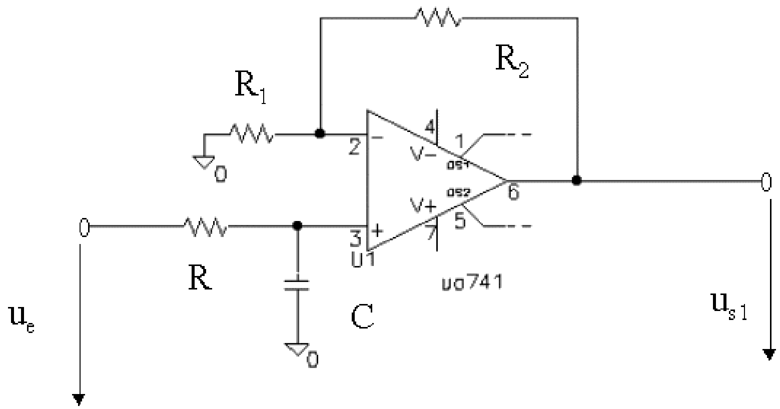


Ejemplos



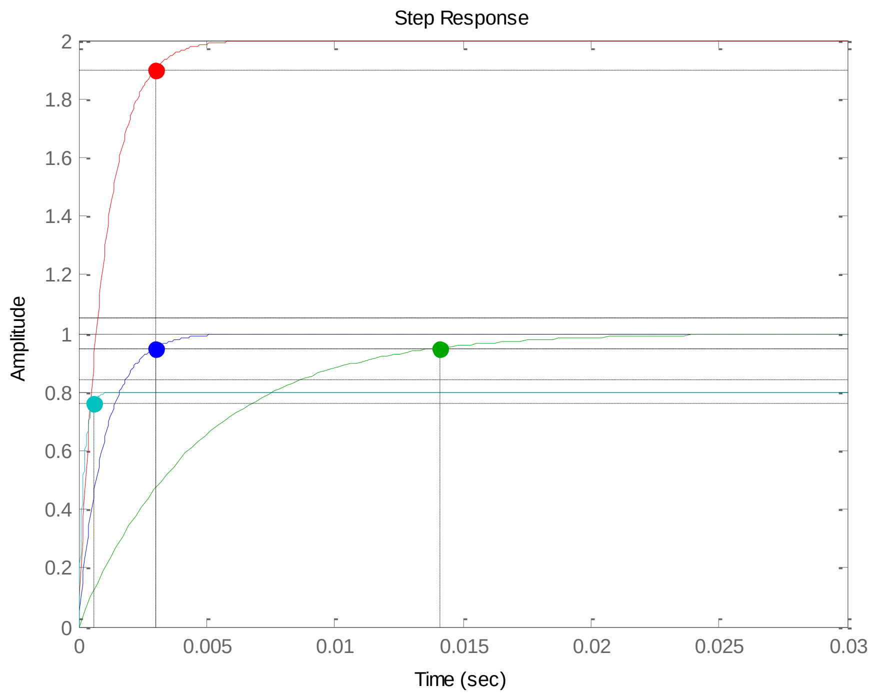
R=100k ó 470k C=10nF



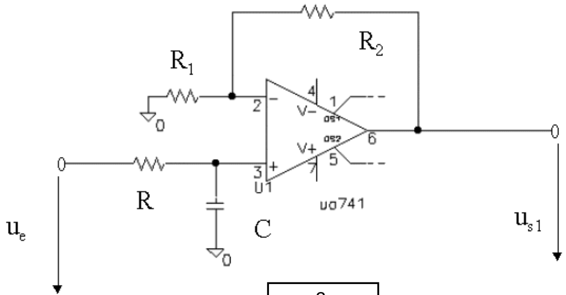


R=100k C=10nF

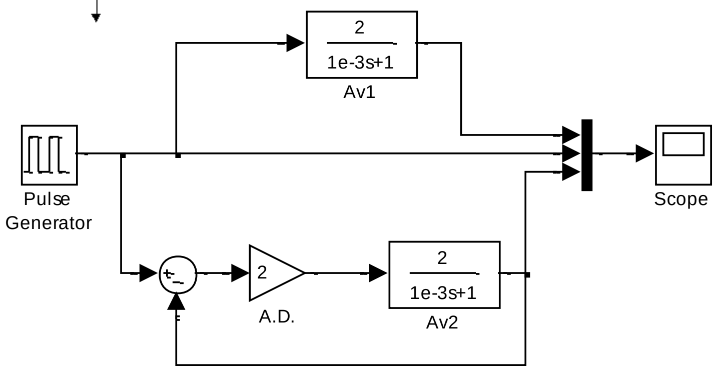
R_1=33k, R_2=33k
R_3=33k, R_4=68k



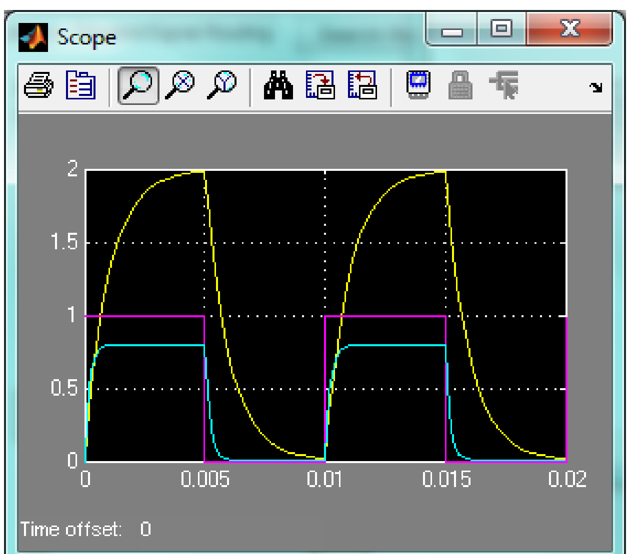
Ejemplos



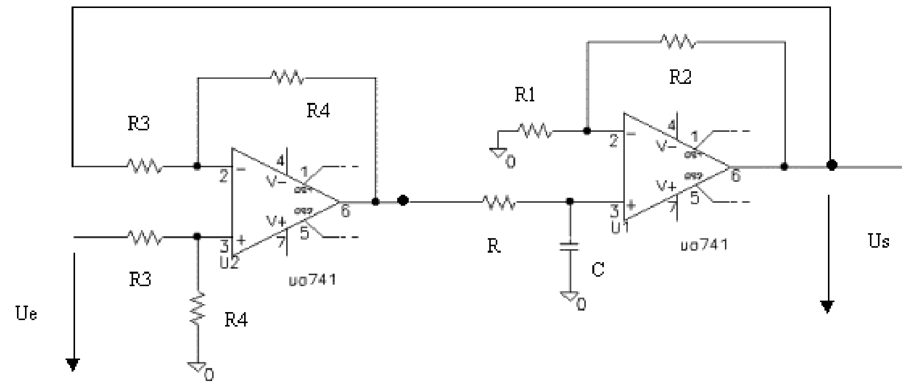
R1
R2
R
C
ue
us1
U1
uo741



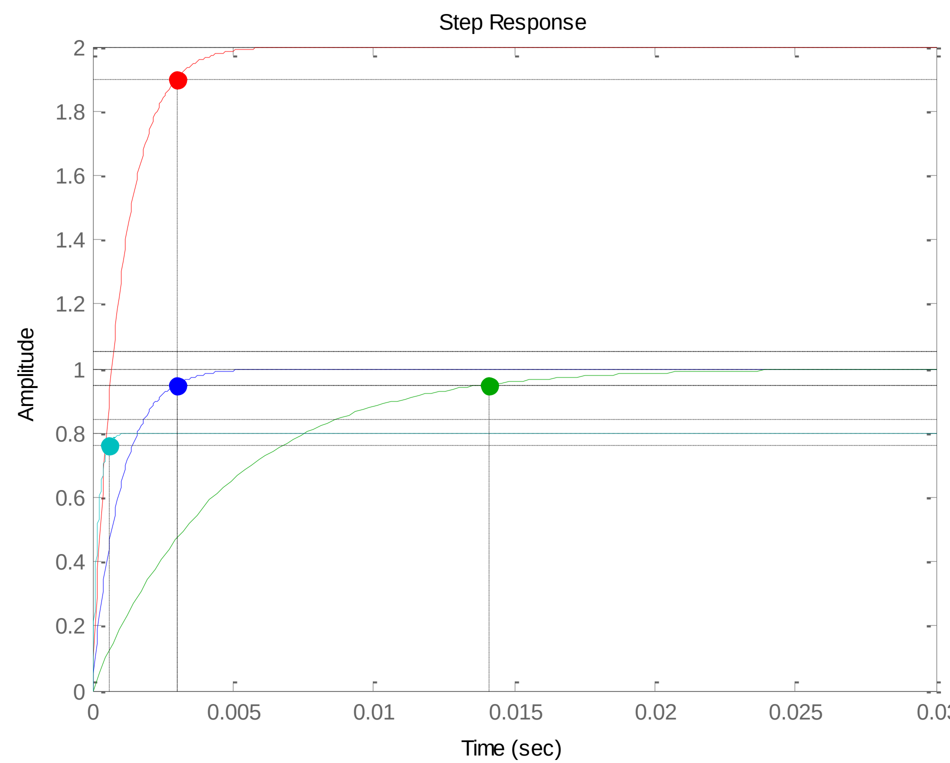
Pulse Generator
2
1e-3s+1
Av1
Scope
2
A.D.
2
1e-3s+1
Av2



Scope
2
1.5
1
0.5
0
0
0.005
0.01
0.015
0.02
Time offset: 0



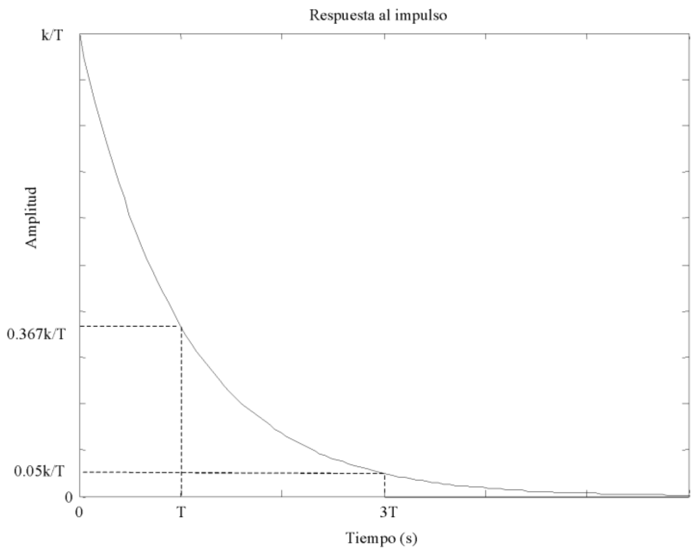
R3
R4
R3
R4
R1
R2
R
C
U2
U1
uo741
uo741
Ue
Us



Step Response
Amplitude
Time (sec)
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.005
0.01
0.015
0.02
0.025
0.03

Respuesta temporal ante el impulso

- Analítica & transformadas de Laplace



$$Y(s) = \frac{k}{1+sT}$$

$$\boxed{Y(s) = G(s) = \frac{k}{1+sT} \quad \Rightarrow \quad y(t) = \frac{k}{T} e^{-t/T} = g(t)}$$

$$\boxed{y_{escalon}(t) = k\left(1 - e^{-t/T}\right) \qquad \dot{y}_{escalon}(t) = \frac{k}{T} e^{-t/T} = g(t)}$$

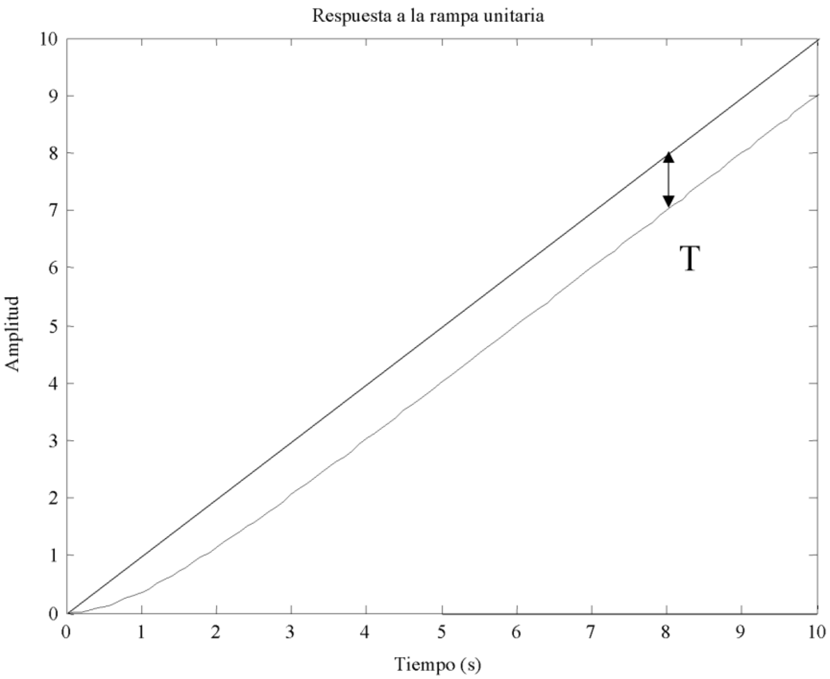
$$y(t = T) = \frac{k}{T} e^{-1} = 0.367 \frac{k}{T} \qquad y(t = 3T) = \frac{k}{T} e^{-3} = 0.05 \frac{k}{T}$$

Valor inicial: $y(t \to 0) = \frac{k}{T}$ $\quad \lim_{s \to \infty} s \cdot 1 \cdot \frac{k}{1+sT} = \frac{k}{T}$

Valor final: $y(t \to \infty) = 0$ $\quad \lim_{s \to 0} s \cdot 1 \cdot \frac{k}{1+sT} = 0$

Respuesta temporal ante la rampa

- Analítica & transformadas de Laplace



$$Y(s) = \frac{1}{s^2}\frac{k}{1+sT} = \frac{a_2}{s^2} + \frac{a_1}{s} + \frac{k_1}{s+\frac{1}{T}}$$

$$\boxed{y(t) = k\left(t - T + Te^{-t/T}\right)}$$

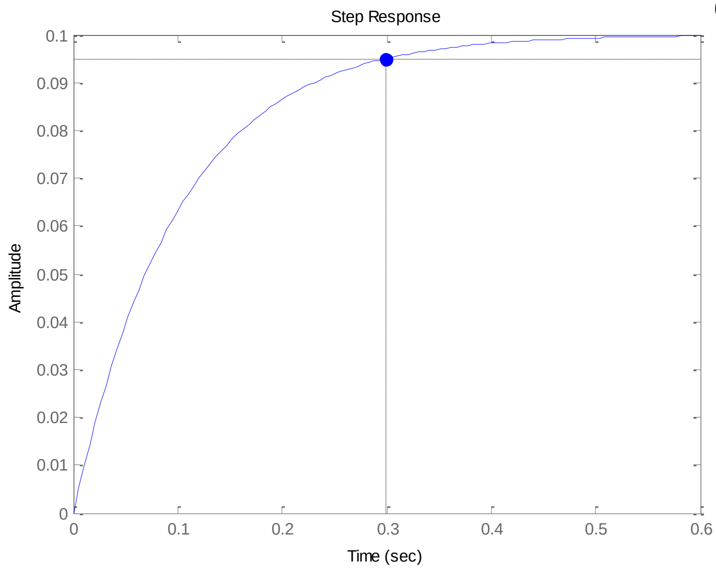
$$y_{rampa}(t) = \int_0^t y_{escalon}(\tau)d\tau = \int_0^t k\left(1-e^{-\tau/T}\right)d\tau = k\left[\tau + Te^{-\tau/T}\right]_0^t$$

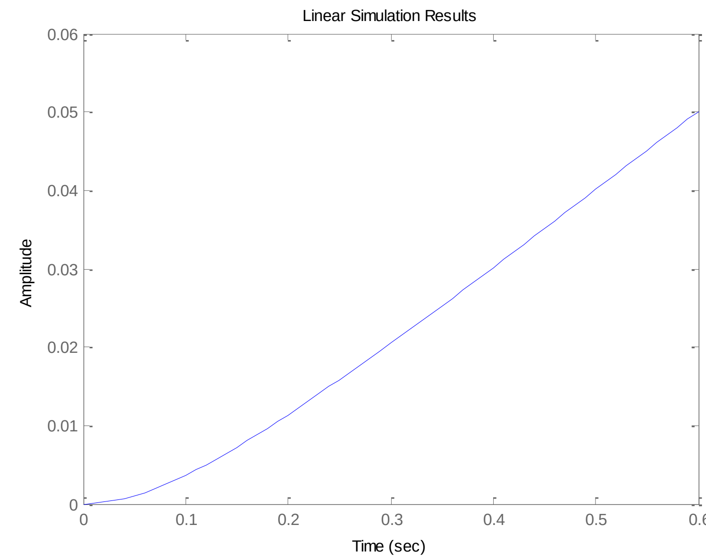
$$\boxed{y_{rampa}(t) = k\left(t + Te^{-t/T} - T\right)}$$

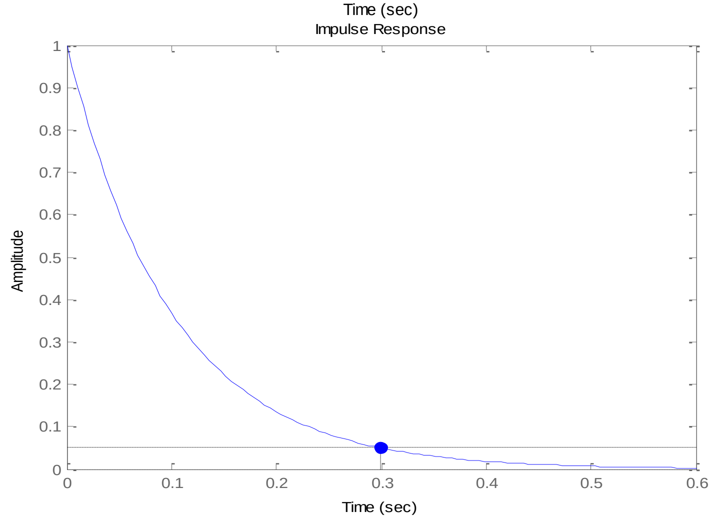
Ejercicio 6.1

Dibujar aproximadamente, la respuesta al impulso, escalón y rampa del sistema cuya FDT es:

$$G(s) = \frac{1}{s+10}$$





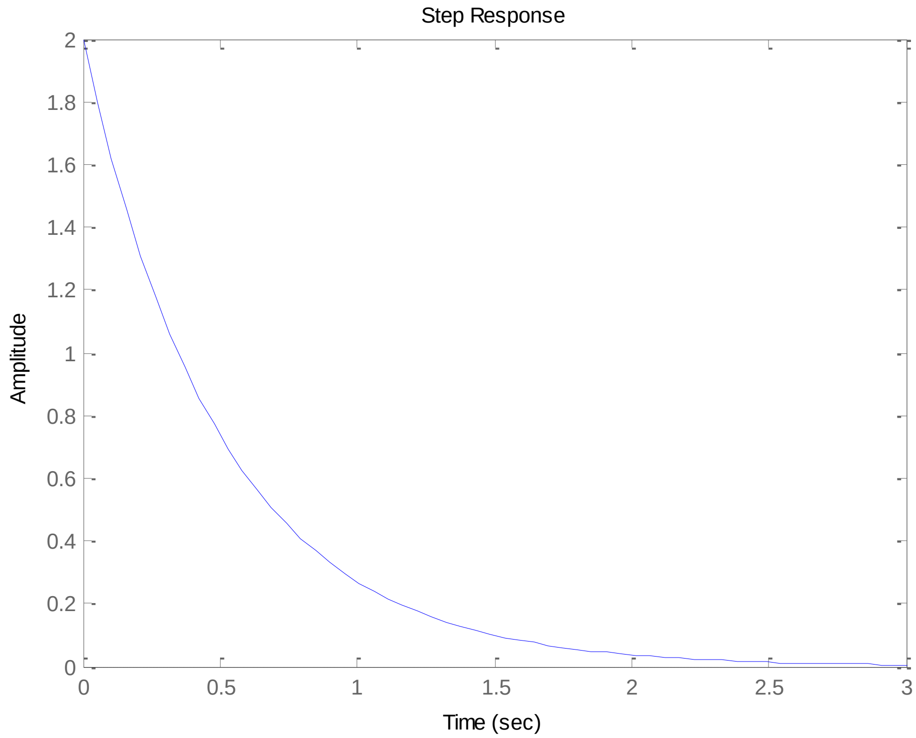



```
>>g1=tf(1,[1 10])
>>step(g1)
>>impulse(g1)
>>t=0:0.001:.6;
>>lsim(g1,t,t)
>>ltiview(g1)
```

Ejercicio 6.2

Dibujar la respuesta al escalón del sistema de:

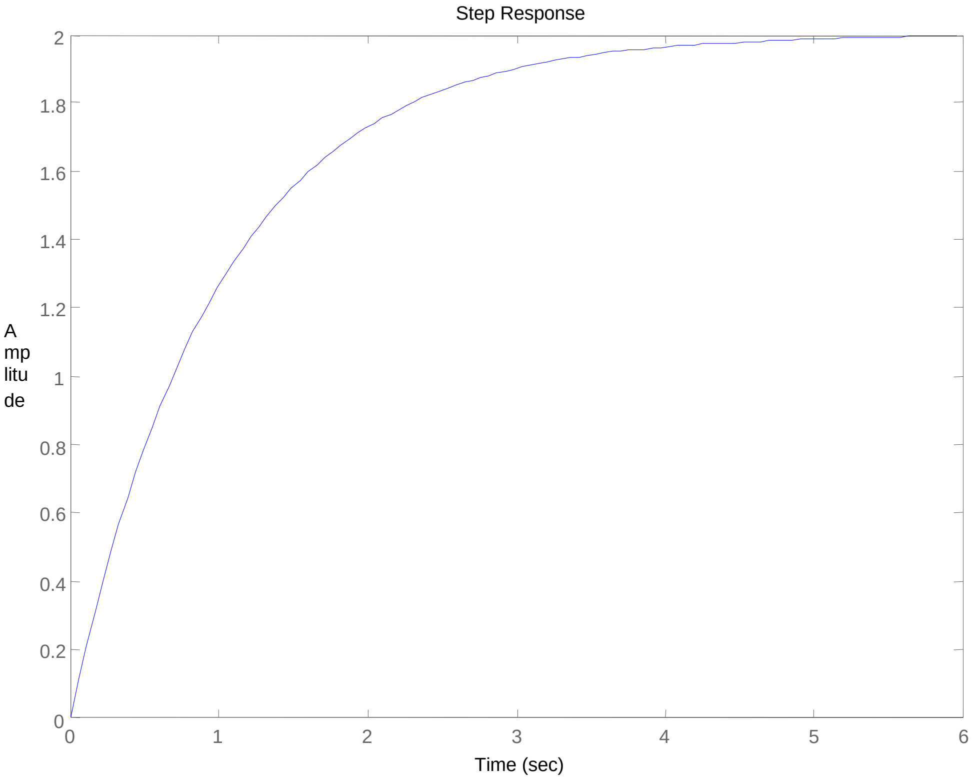
$$G(s) = \frac{2s}{s+2}$$



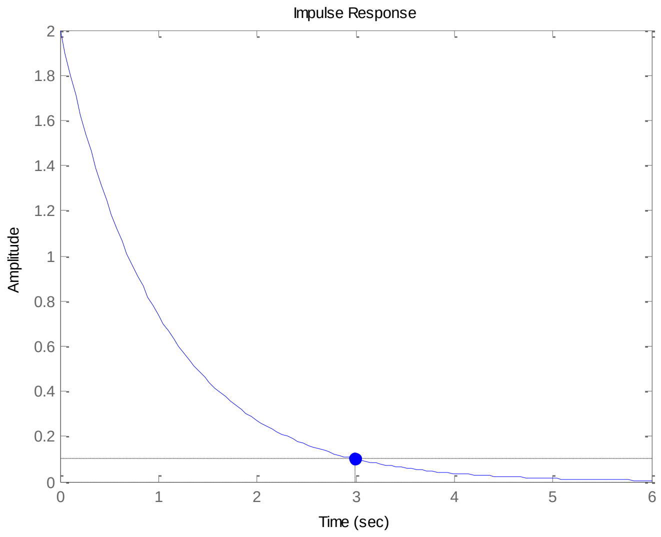
>>g2=tf([2 0],[1 2])
>>step(g2)

Ejercicio 6.3

La figura representa la respuesta al escalón de un sistema de FDT desconocida. Obtener la respuesta del sistema ante una entrada en impulso:



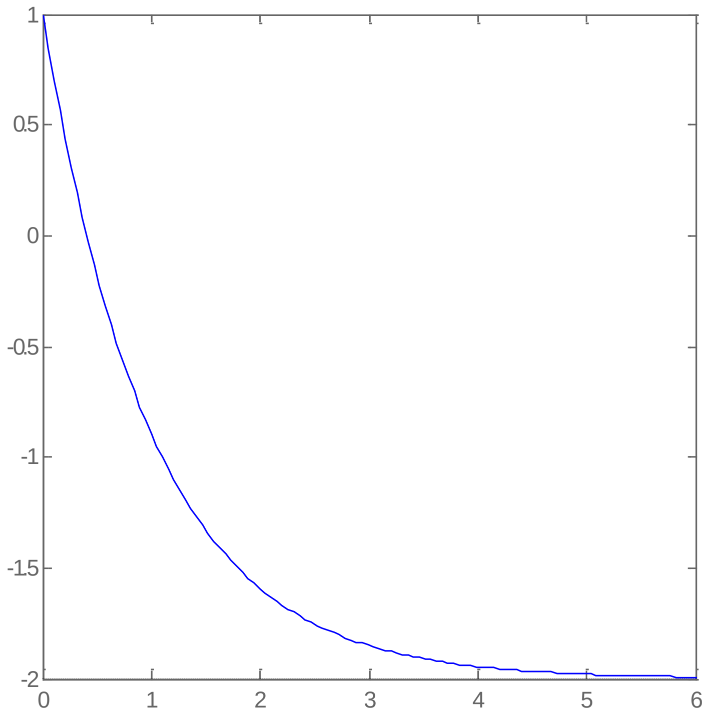
[>>g3=tf(2,[1 1])](#)
[>>step(g3)](#)



Ejercicio

Dibujar la respuesta al escalón del sistema de:

$$G(s) = \frac{s-2}{s+1}$$

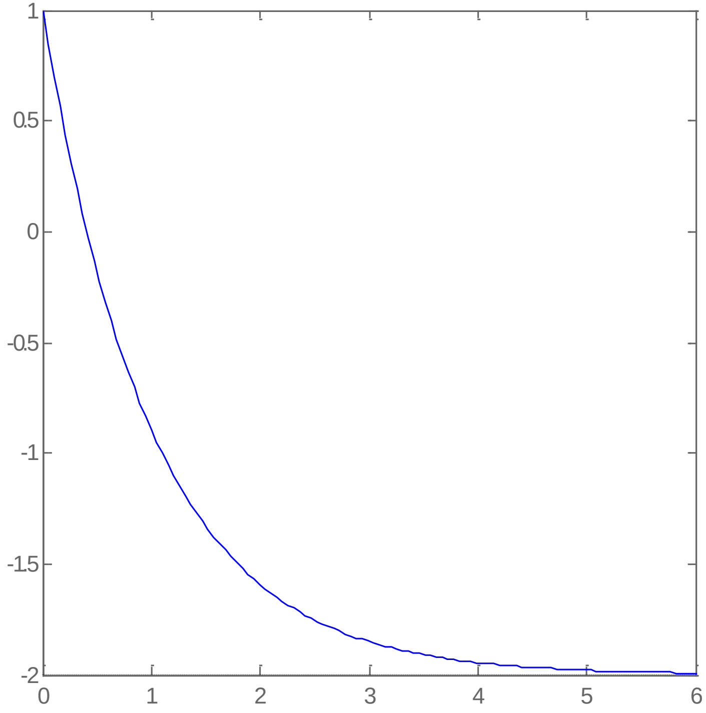



```
>>g2=tf([1 -2],[1 1])
>>step(g3)
```

Ejercicio

Dibujar la respuesta al escalón del sistema de:

$$G(s) = \frac{s-2}{s+1}$$

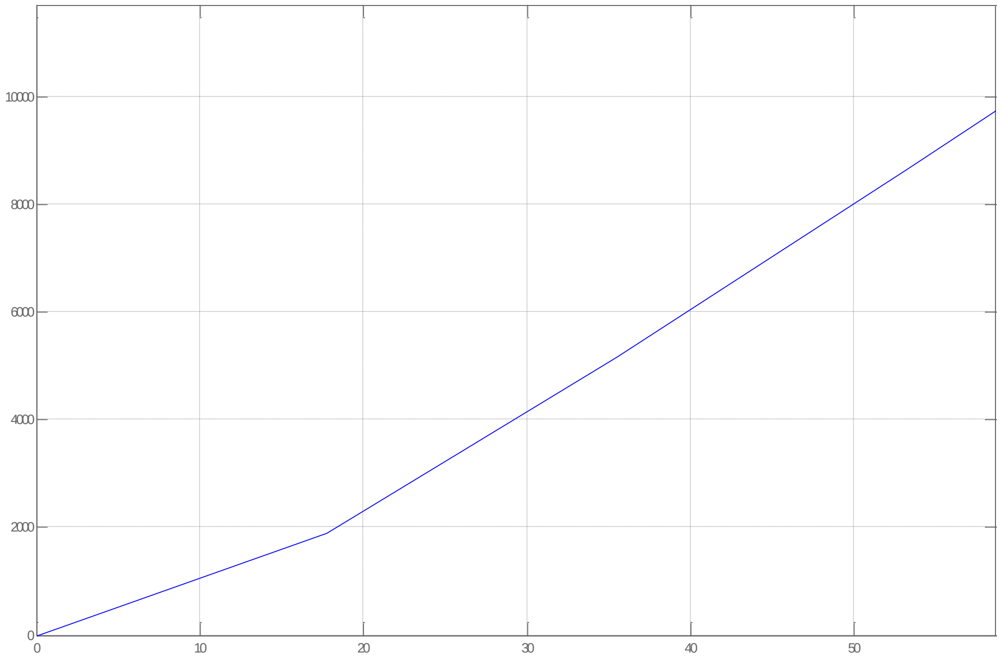


>>g2=tf([1 -2],[1 1])
>>step(g3)

Ejercicio

Dibujar la respuesta al escalón del sistema de:

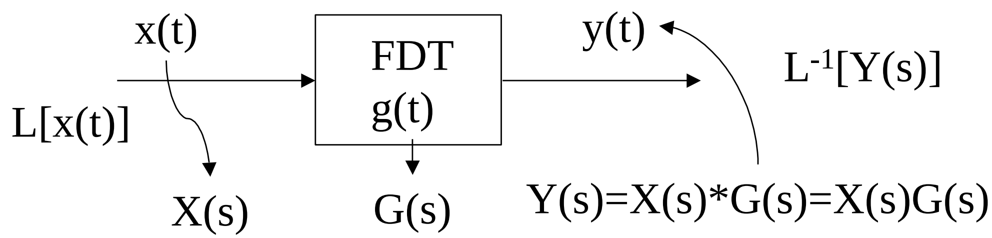
$$G(s) = \frac{s+20}{s^2+0.1s}$$



>>g2=tf([1 20],[1 .1 0])
>>step(g3)

Análisis temporal de sistemas de segundo orden

- ## Modelo

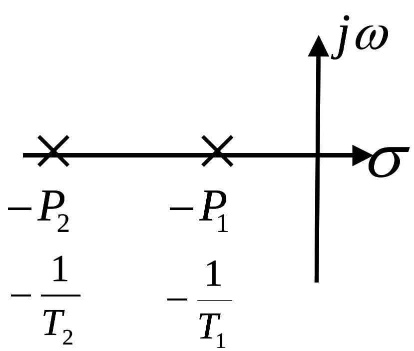


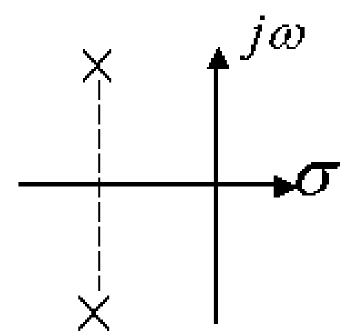
$$\boxed{a_0 y + a_1 \dot{y} + a_2 \ddot{y} = b_0 x + b_1 \dot{x} + b_2 \ddot{x}}$$

$$\boxed{G(s) = \frac{Y(s)}{X(s)} = \frac{b_0 + b_1(s) + b_2 s^2}{a_0 + a_1(s) + a_2 s^2}}$$

- ## Sistema de segundo orden simple

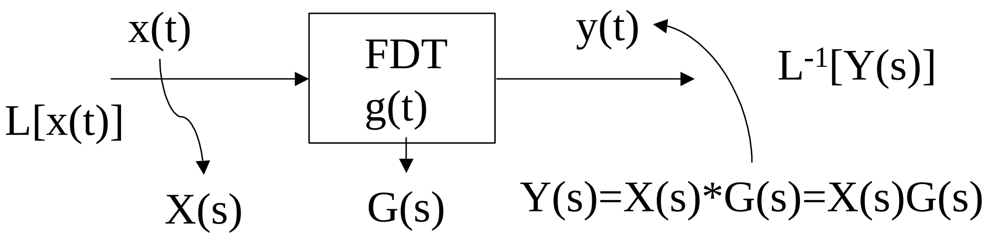
$$G(s) = \frac{b_0}{a_0 + a_1 s + a_2 s^2}$$

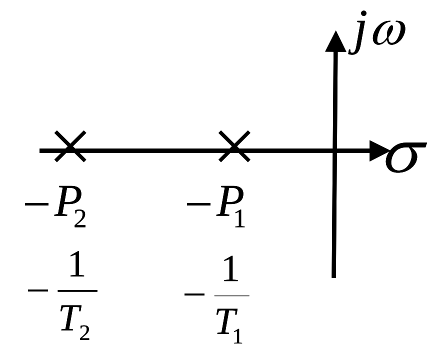




Sistemas sobre-amortiguados de segundo orden

- Polos reales



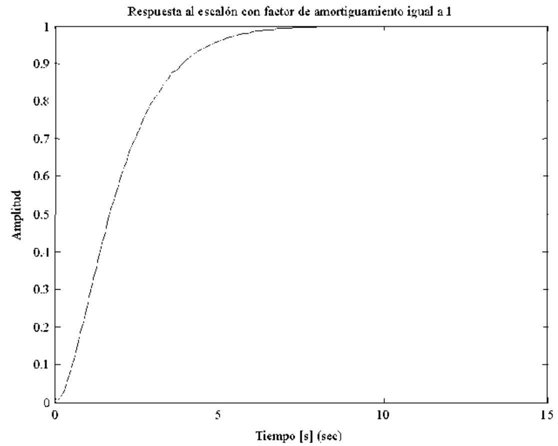


$$G(s)=\frac{b_0}{(s+p_1)(s+p_2)}$$

- Respuestas al escalón unitario

$$Y(s)=\frac{1}{s}\frac{b_0}{(s+p_1)(s+p_2)}=\frac{k_1}{s}+\frac{k_2}{s+p_1}+\frac{k_3}{s+p_2}$$

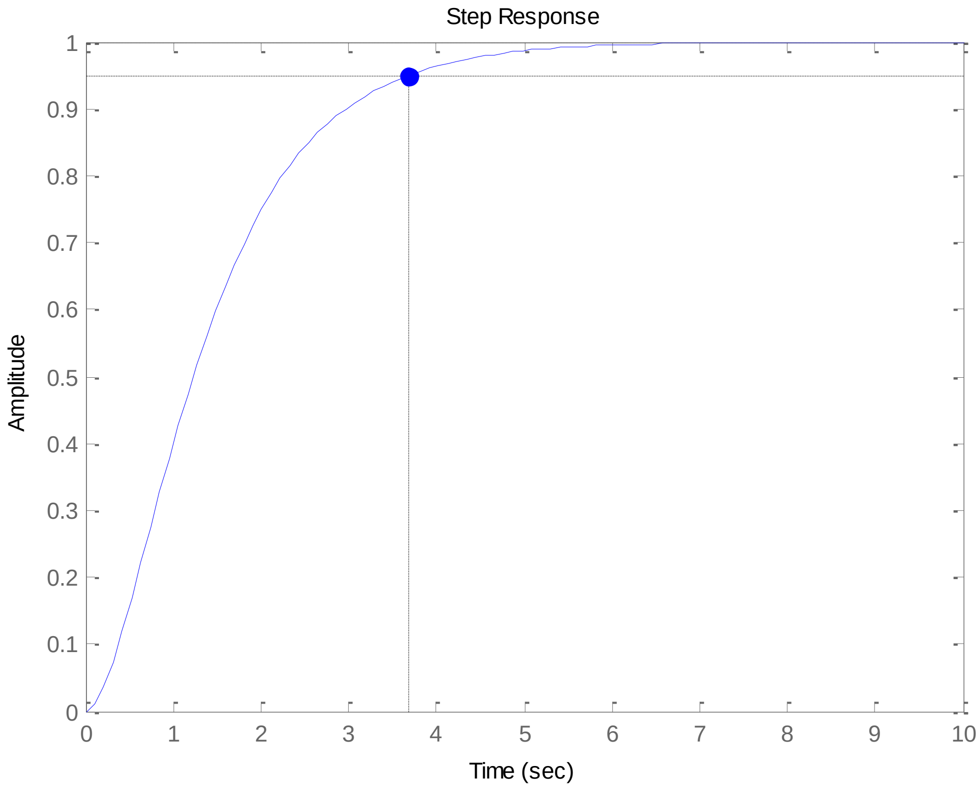
$$y(t)=k_1+k_2e^{-p_1t}+k_3e^{-p_2t}$$



Ejercicios

Dibujar la respuesta al escalón del sistema de:

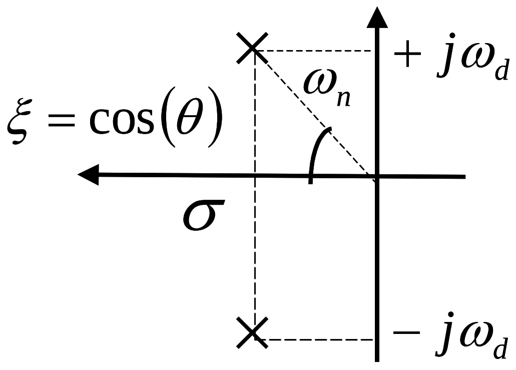
$$G_5(s) = \frac{2}{(s+1)(s+2)} \qquad G_6(s) = \frac{2}{(s+1)(s-2)}$$




```
>>g5=tf(2,poly([-1 -2]))
>>step(g5)
```

Sistemas sub-amortiguados de segundo orden

- Polos complejos y conjugados
  - Parámetros: k,ω_n y ξ



$$G(s) = \frac{k}{\left(\frac{s}{\omega_n}\right)^2 + 2\xi\left(\frac{s}{\omega_n}\right) + 1} = \frac{k\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\frac{-2\xi\omega_n \pm \sqrt{(2\xi\omega_n)^2 - 4\omega_n^2}}{2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

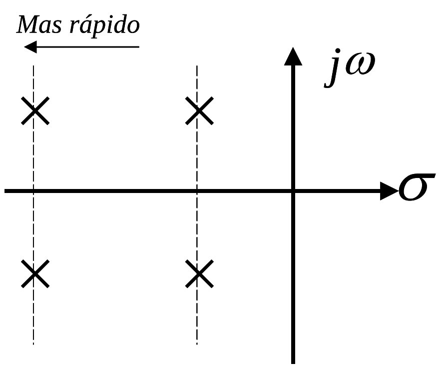
$\boxed{\sigma = \xi\omega_n}$ $\boxed{\omega_n^2 = \sigma^2 + \omega_d^2 \qquad 0 \le |\xi| \le 1}$ $\boxed{\xi = \cos\theta \qquad 0 \le |\xi| \le 1}$

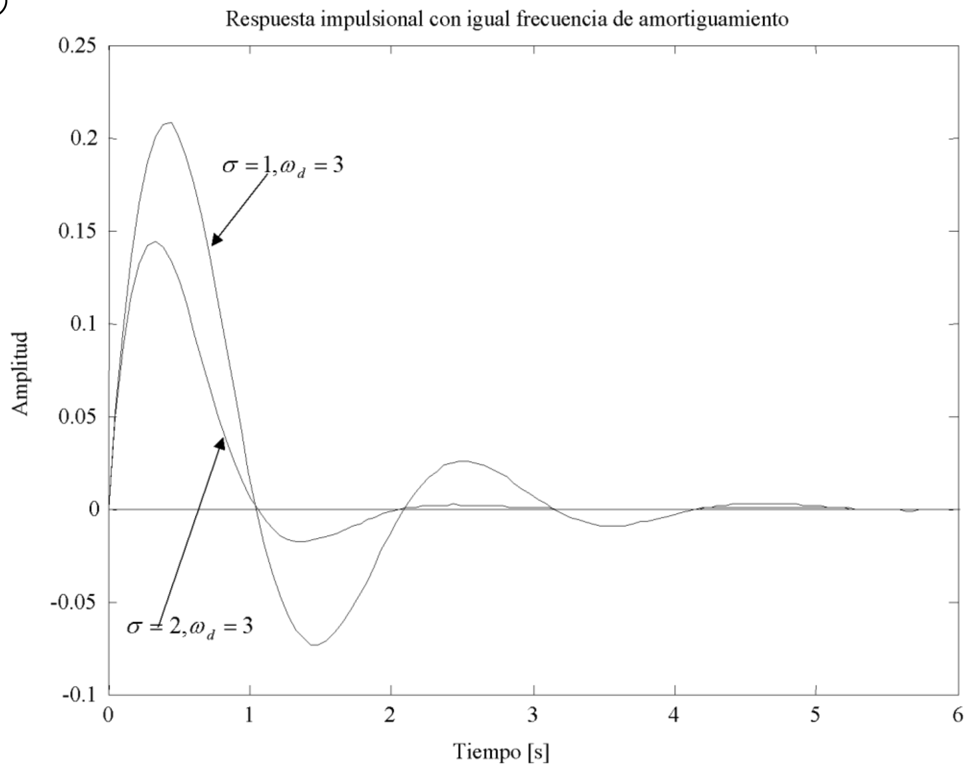
Respuesta al impulso de un sistema de segundo orden sub-amortiguado

$$\boxed{G(s)=\frac{k\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}=\frac{k\omega_n^2}{(s+\sigma-j\omega_d)(s+\sigma+j\omega_d)}=\frac{k_1}{(s+\sigma-j\omega_d)}+\frac{k_2}{(s+\sigma+j\omega_d)}}$$

$$\left.\begin{aligned}
k_1&=\left[(s+\sigma-j\omega_d)G(s)\right]_{s=-\sigma+j\omega_d}=\frac{k\omega_n^2}{2j\omega_d}=\frac{k\omega_n}{2j\sqrt{1-\xi^2}}\\
k_2&=\left[(s+\sigma+j\omega_d)G(s)\right]_{s=-\sigma-j\omega_d}=\frac{k\omega_n^2}{-2j\omega_d}=\frac{-k\omega_n}{2j\sqrt{1-\xi^2}}
\end{aligned}\right\}$$

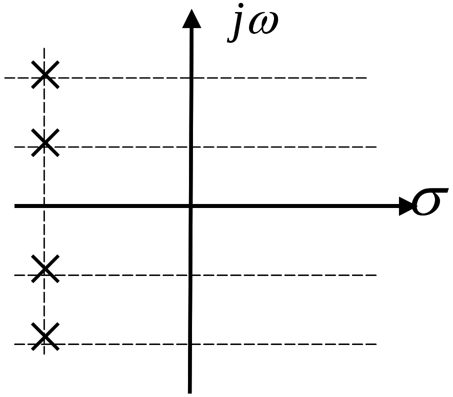
$$\boxed{g(t)=\frac{k\omega_n}{\sqrt{1-\xi^2}}e^{-\sigma t}\left(\frac{e^{+j\omega_d t}-e^{-j\omega_d t}}{2j}\right)=\frac{k\omega_n}{\sqrt{1-\xi^2}}e^{-\sigma t}sen(\omega_d t)}$$

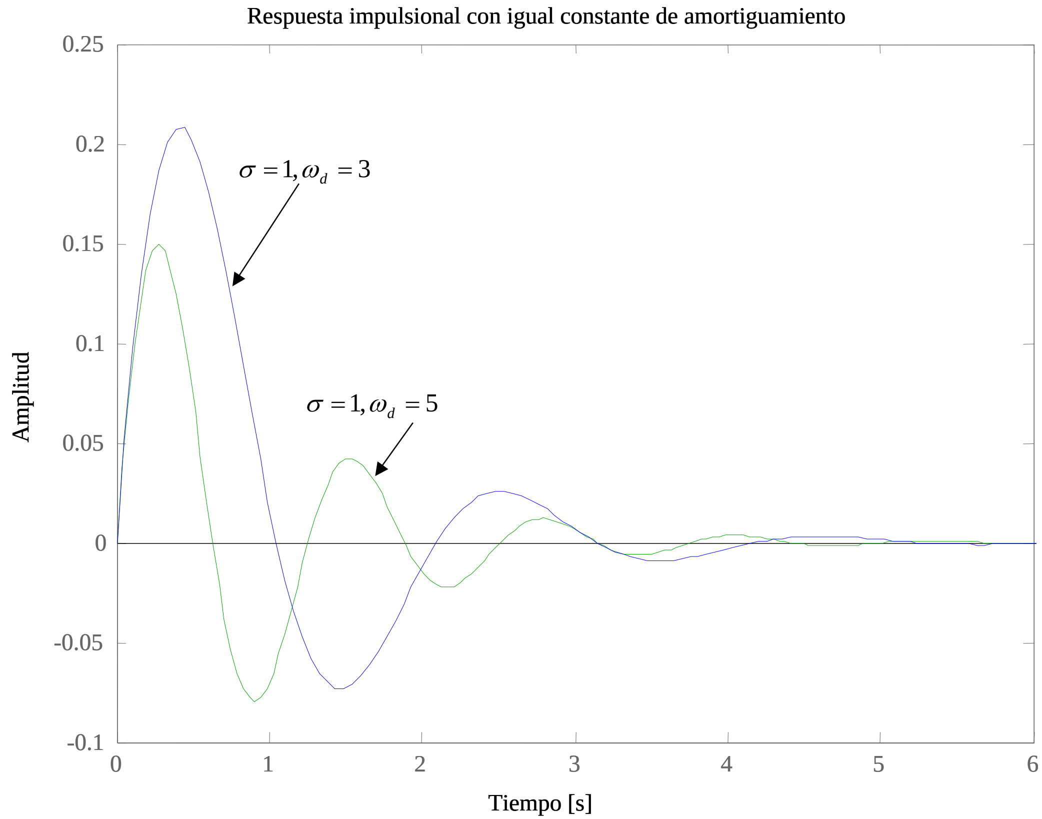




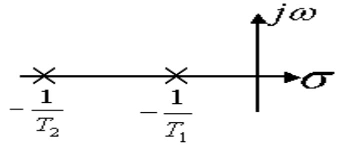
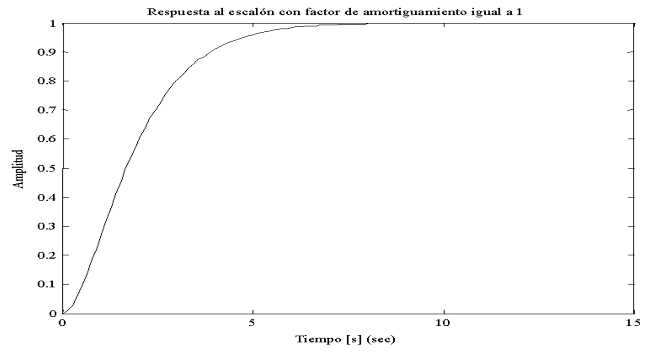
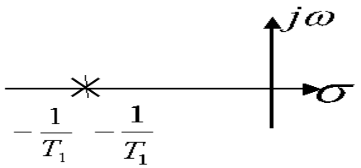
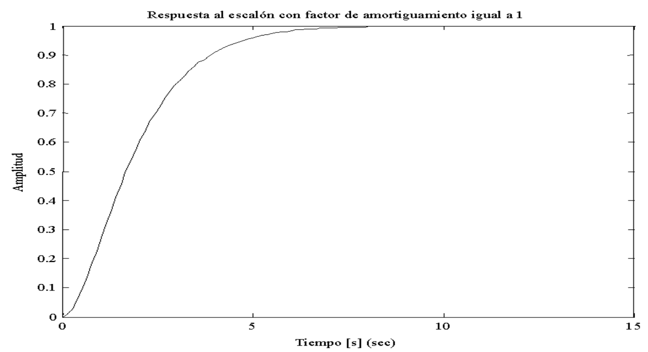
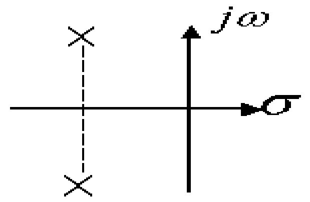
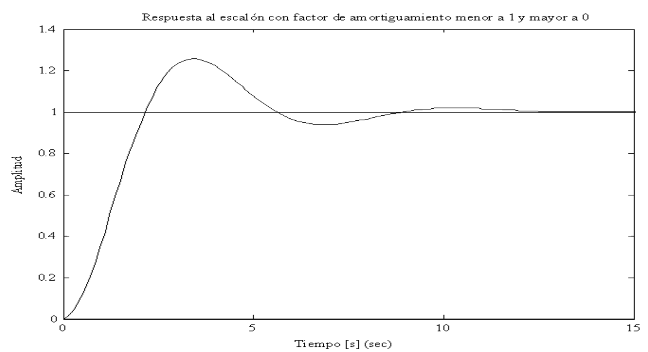
Respuesta al impulso de un sistema de segundo orden sub-amortiguado

$$g(t)=\frac{k\omega_n}{\sqrt{1-\xi^2}}e^{-\sigma t}\left(\frac{e^{+j\omega_d t}-e^{-j\omega_d t}}{2j}\right)=\frac{k\omega_n}{\sqrt{1-\xi^2}}e^{-\sigma t}sen(\omega_d t)$$





Respuesta al escalón en sistemas de 2º

Situación del polo	Respuesta al escalón	Sistema
		Sobre Amortiguado $\xi > 1$
		Críticamente amortiguado $\xi = 1$
		Sub amortiguado $0 < \xi < 1$

Respuesta al escalón en sistemas de 2º

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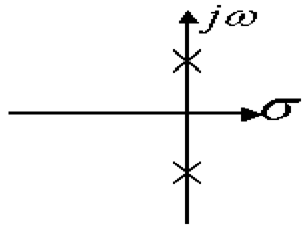
<td>
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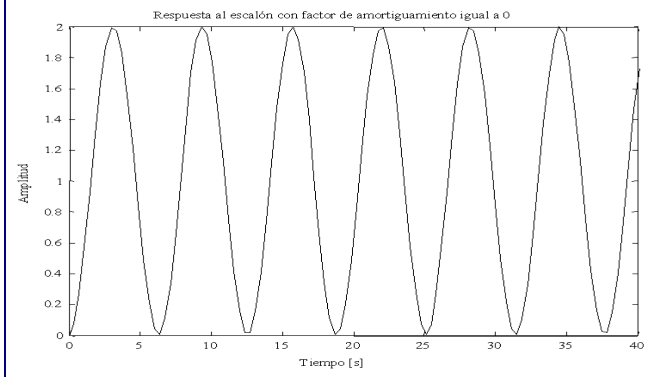
<td>
</td>
<td>Críticamente estable
$\xi=0$</td>
</tr>
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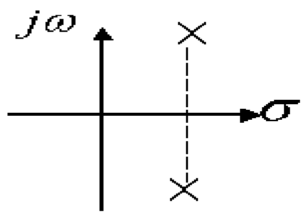
<td>
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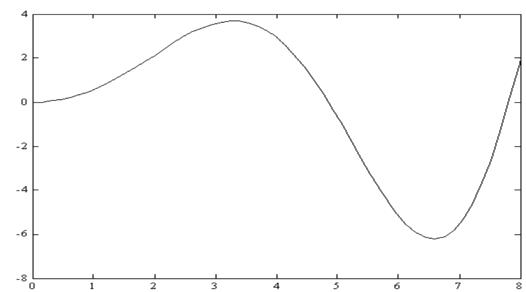
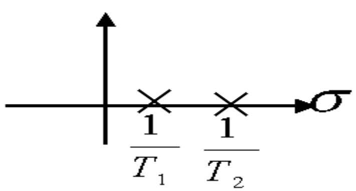
<td></td>
<td>INESTABLE
$-1<\xi<0$</td>
</tr>
<tr>
<td>
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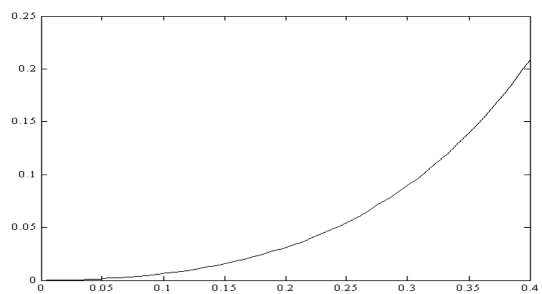
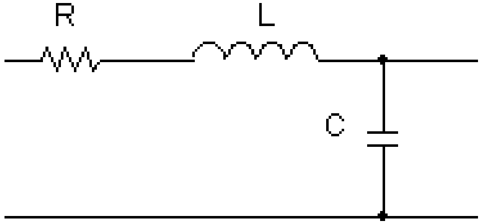
<td></td>
<td>INESTABLE
$\xi<-1$</td>
</tr>
</table>

Ejemplo

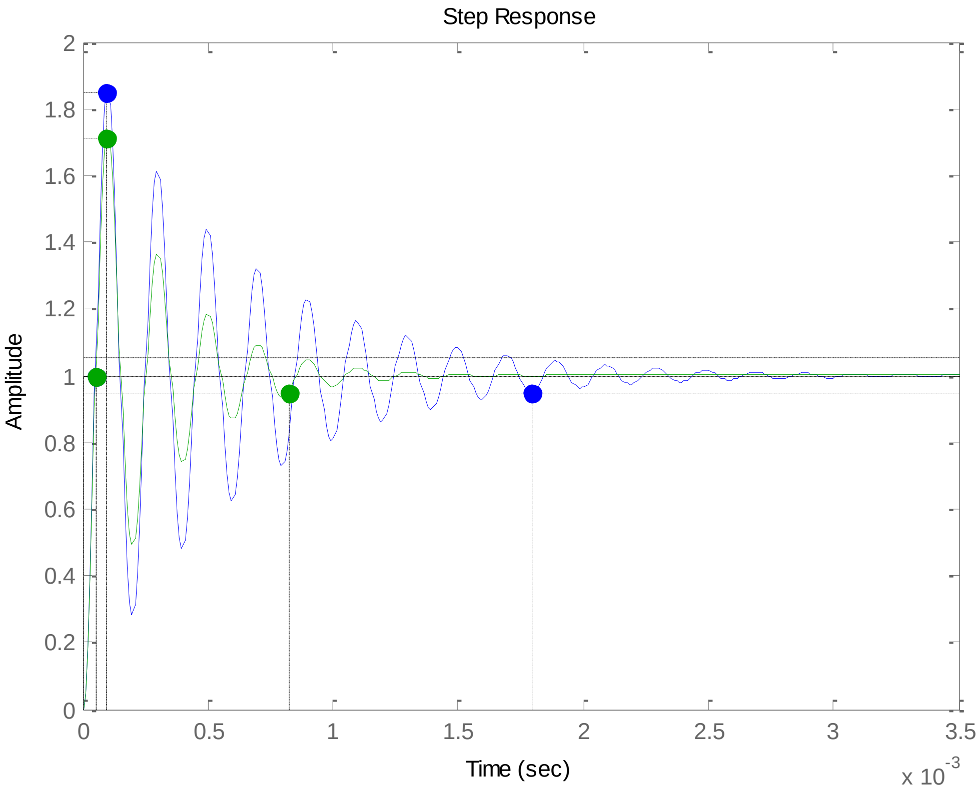
- Respuesta al escalón unitario
  - C = 10 nF, L = 100 mH y R = 330 Ω / 680 Ω



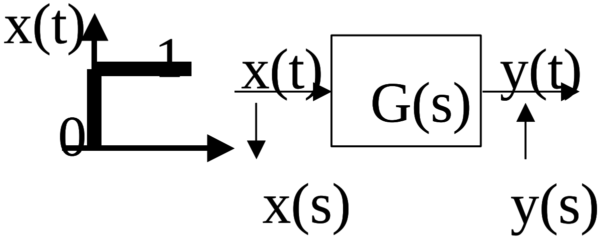
$$\frac{u_s(t)}{u_e(t)} = \frac{1}{LCs^2 + RCs + 1}$$

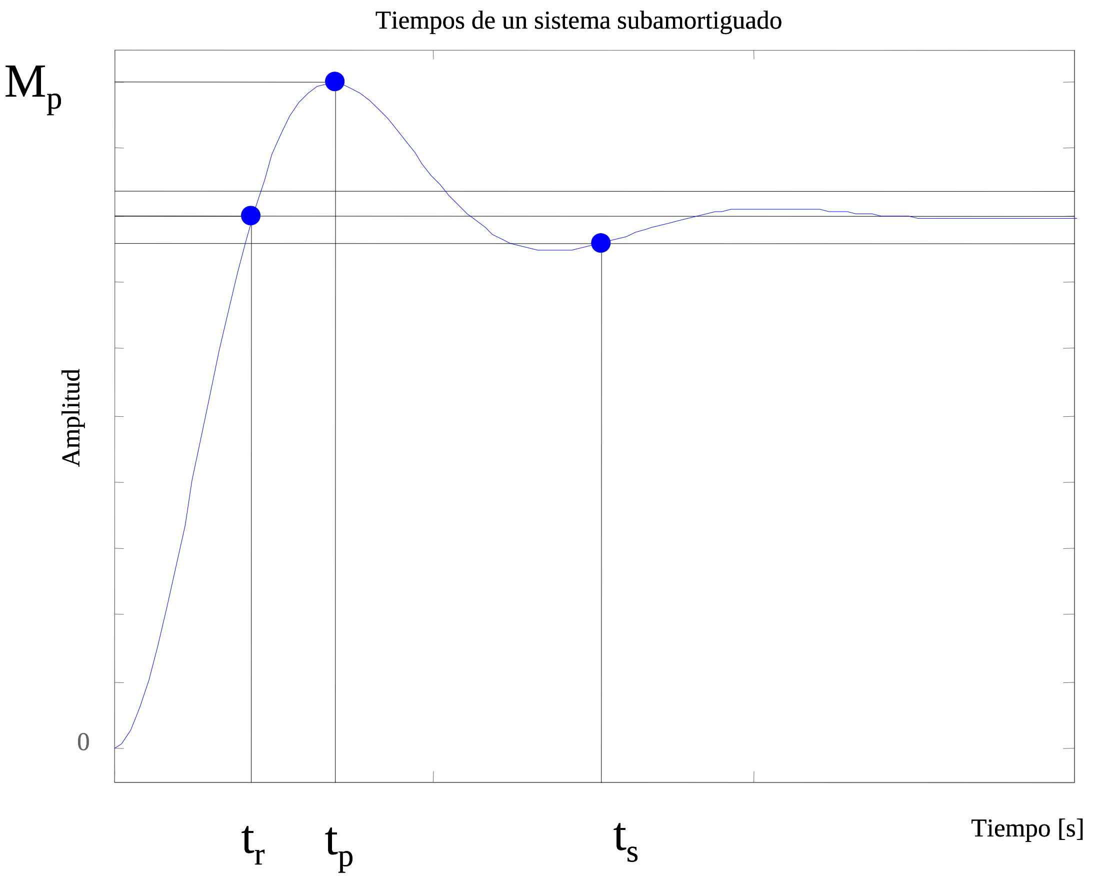
$$\omega_n = \frac{1}{\sqrt{LC}} = 31623[rad/s]$$

$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}} = 0.052 \quad 0.1$$



Respuesta en escalón en sistemas sub-amortiguados



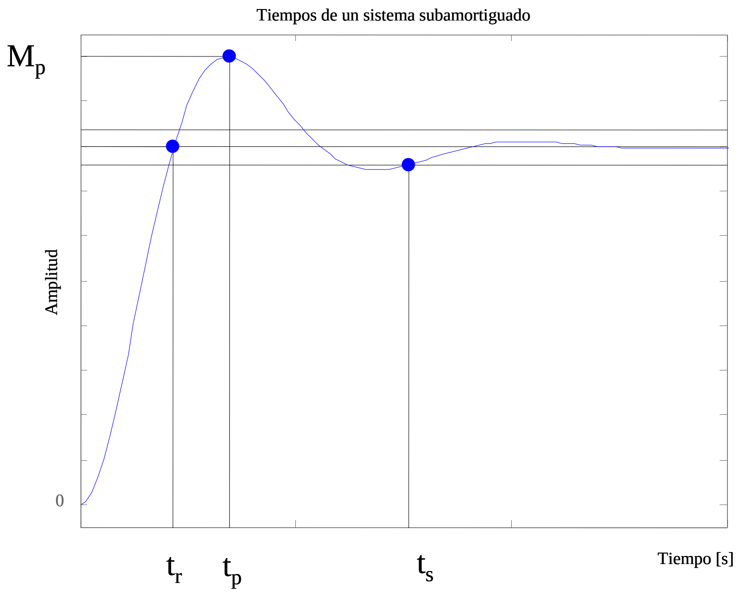


$$y(t) = k\left(1 - \frac{e^{-\sigma t}}{\sqrt{1-\xi^2}} sen(\omega_d t + \theta)\right)$$

$$0 < \xi < 1$$

Tiempo de establecimiento

- t_s : valor de tiempo que el sistema necesita en alcanzar un error del 5% ó 2%, según criterio, del valor final del régimen permanente.



$$y(t) = k\left(1 - \frac{e^{-\sigma t}}{\sqrt{1-\xi^2}}\, sen(\omega_d t + \theta)\right)$$

$$\downarrow$$

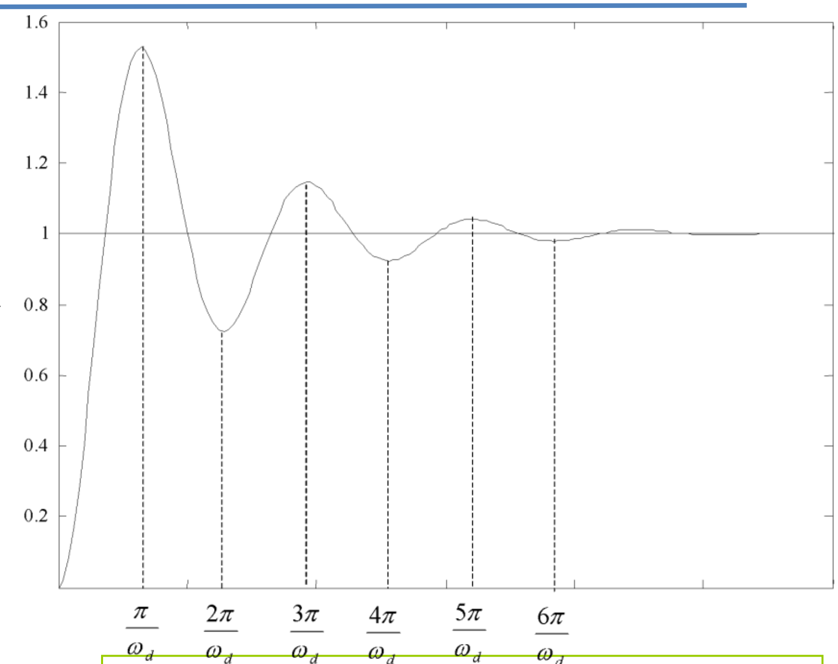
$$\frac{e^{-\sigma t_s}}{\sqrt{1-\xi^2}} \cong 0.05 = e^{-\pi} \longrightarrow$$

$$\xi << 1 \rightarrow t_s \cong \frac{\pi}{\sigma}$$

$$\uparrow \sigma \rightarrow \downarrow t_s$$

Tiempo de pico

- t_p: intervalo de tiempo en darse la máxima amplitud de salida (sólo es válido si el factor de amortiguamiento está entre 0 y 0.7). En caso contrario, no habrá sobreoscilación y no tiene sentido este parámetro.



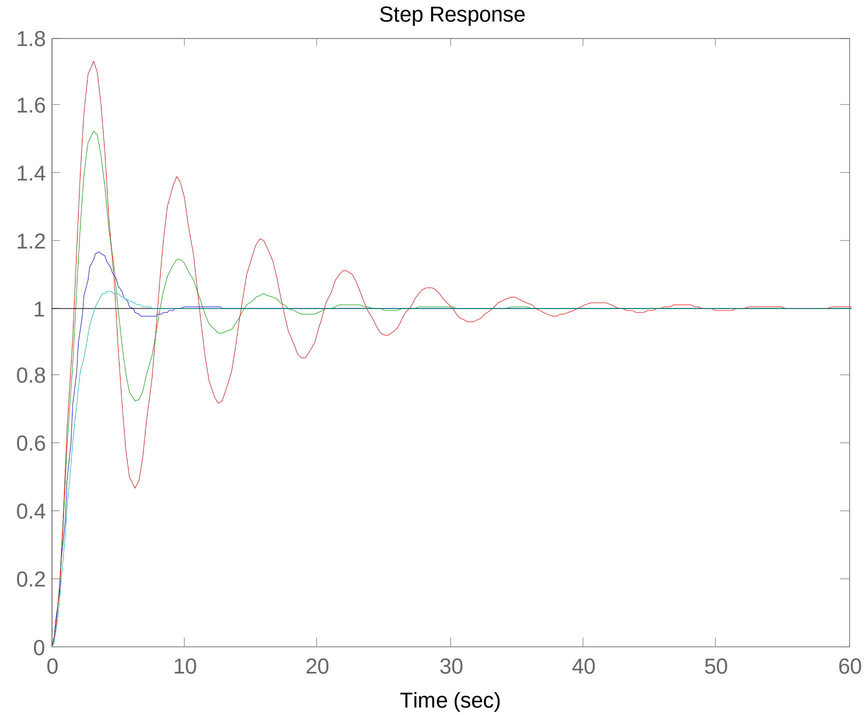
$$\boxed{y(t) = k\left(1 - \frac{e^{-\sigma t}}{\sqrt{1-\xi^2}}\, sen(\omega_d t + \theta)\right)}$$

$$\frac{dy(t)}{dt} = 0 = -k\left(\frac{(-\sigma)\cdot e^{-\sigma t_p}}{\sqrt{1-\xi^2}}\, sen(\omega_d t_p + \theta) + \frac{e^{-\sigma t_p}}{\sqrt{1-\xi^2}} \cos(\omega_d t_p + \theta).\omega_d\right) \qquad tg(\omega_d t_p + \theta) = \frac{\omega_d}{\sigma} = \frac{\omega_n\sqrt{1-\xi^2}}{\xi\omega_n} = tg(\theta)$$

$$\boxed{\begin{array}{l} \omega_d t_p = \pi \rightarrow t_p = \dfrac{\pi}{\omega_d} \\ \uparrow \omega_d \rightarrow \downarrow t_p \end{array}}$$

Sobreoscilación

- M_p: Valor de pico máximo de la salida ponderado con el valor final. Sólo sucede si el factor de amortiguamiento está entre 0 y 0.707
- Compromiso entre estabilidad y rapidez (diseño):
  - el factor de amortiguamiento debe estar entre 0.4 y 0.7, lo cual significa una sobreoscilación entre el 12% y el 30%



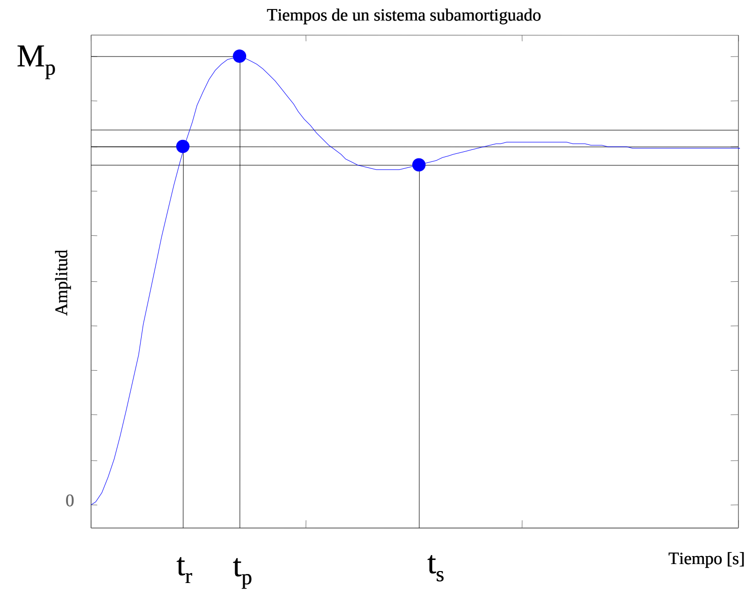
$$M_p = \frac{y_{\max} - y_{rp}}{y_{rp}} = \frac{k\left(1 - \dfrac{e^{-\sigma\pi/\omega_d}}{\sqrt{1-\xi^2}}\, sen(\pi+\theta)\right) - k}{k} = \frac{1 + \left(\dfrac{e^{-\sigma\pi/\omega_d}}{\sqrt{1-\xi^2}}\, sen(\theta)\right) - 1}{1}$$

$$\boxed{\begin{array}{l} M_p = e^{-\pi\sigma/\omega_d} = e^{-\pi/tg(\theta)} \quad ; \quad M_p[\%] = e^{-\pi/tg(\theta)} \cdot 100\,\% \\ \downarrow \xi \rightarrow \uparrow M_p \end{array}}$$

Tiempo de subida

- t_r: el tiempo transcurrido en alcanzar por primera vez el 100% del valor final de la señal de salida

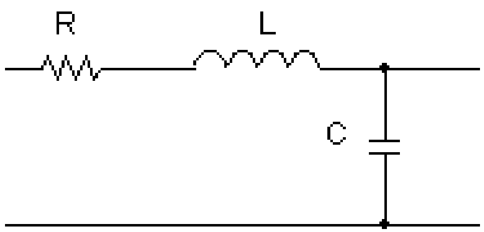
$$\frac{e^{-\sigma t}}{\sqrt{1-\xi^2}}\, sen(\omega_d t_r + \theta) = 0 \rightarrow sen(\omega_d t_r + \theta) = 0$$



$$\omega_d t_r + \theta = \pi \rightarrow t_r = \frac{\pi - \theta}{\omega_d}$$

Ejercicio de la práctica

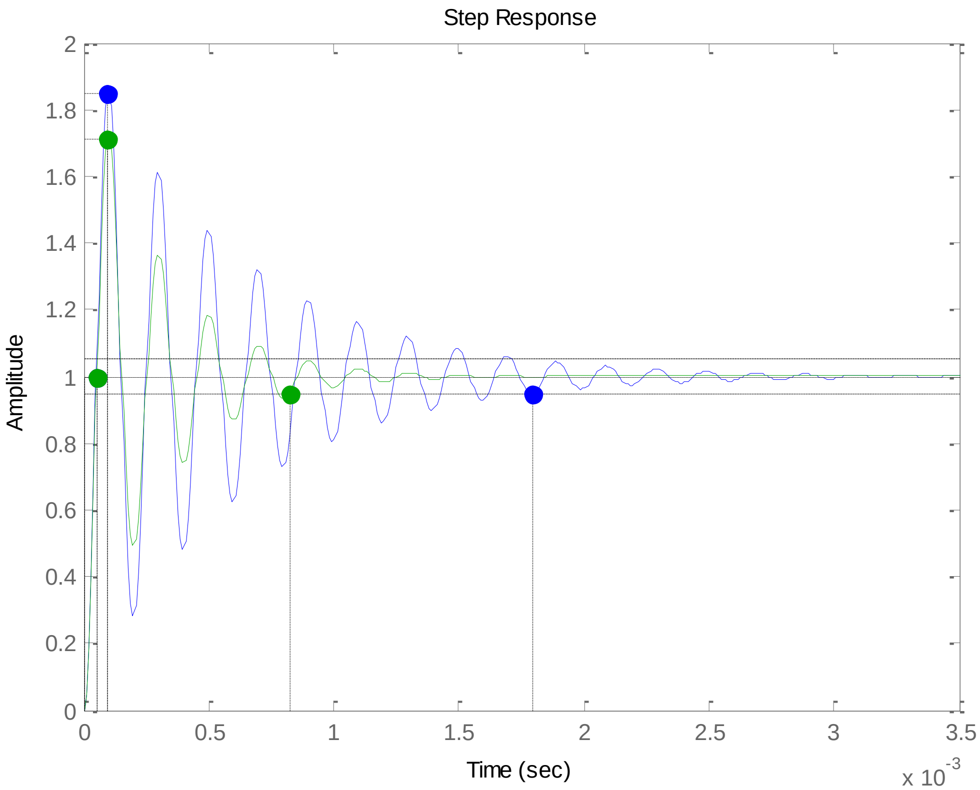
- Respuesta al escalón unitario
  - C = 10 nF, L = 100 mH y R = 330 Ω / 680 Ω



$$\frac{u_s(t)}{u_e(t)} = \frac{1}{LCs^2 + RCs + 1}$$

$$\omega_n = \frac{1}{\sqrt{LC}} = 31623[rad/s]$$

$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}} = 0.052 \quad 0.1$$



$$t_s = 1.9ms \quad t_p = 94\mu s \quad M_p = 85\% \quad (330\Omega)$$

$$t_s = 1ms \quad t_p = 98\mu s \quad M_p = 73\% \quad (680\Omega)$$

Ejercicio 6.4

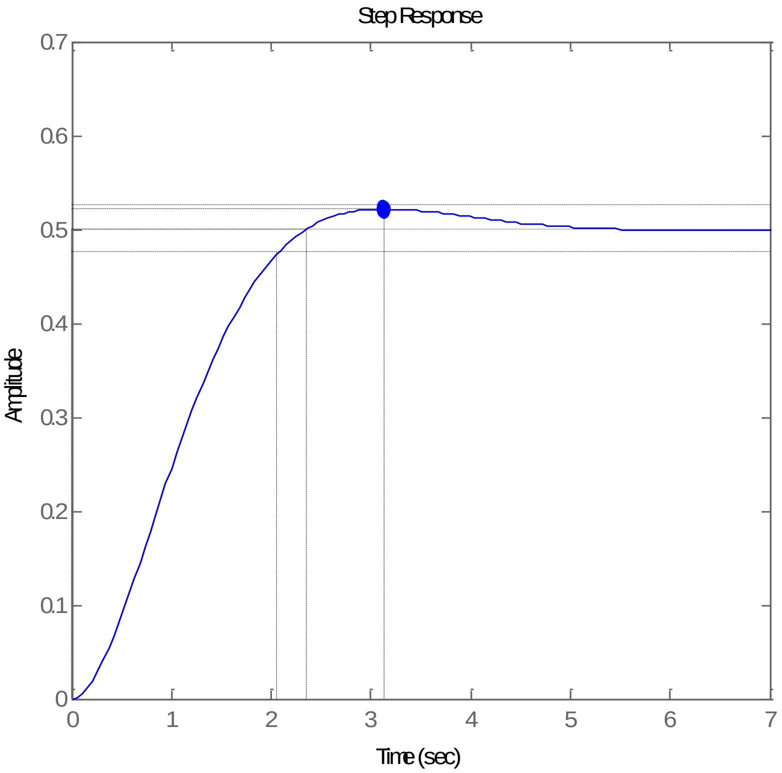
Dibujar la respuesta al escalón del sistema de:

$$G_7(s) = \frac{1}{(s+1+j)(s+1-j)}$$

$$G_8(s) = \frac{1}{(s-1+j)(s-1-j)}$$

$t_s = 3.14s \quad t_p = 3.14s \quad M_p = 5\% \quad t_r = 2.3s$

```
>>g7=tf(1,poly([-1+j -1-j]))
>>step(g7)
```

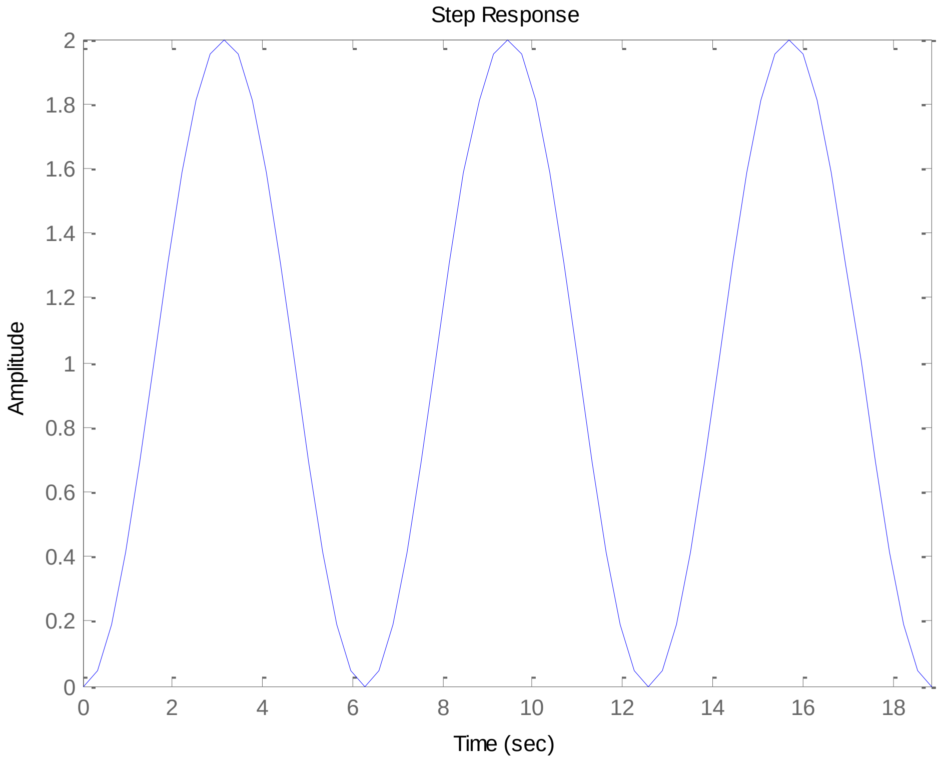


Ejercicio 6.4

Dibujar la respuesta al escalón del sistema de:

$$G_9(s) = \frac{1}{(s^2+1)} \qquad G_{10}(s) = \frac{1}{(s^2-1)}$$

$$t_s = \infty s \quad t_p = 3.14s \quad M_p = 100\% \quad t_r = 1.57s$$

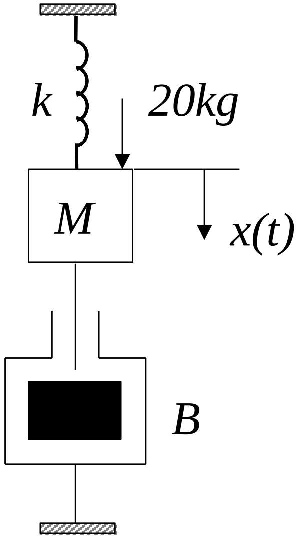


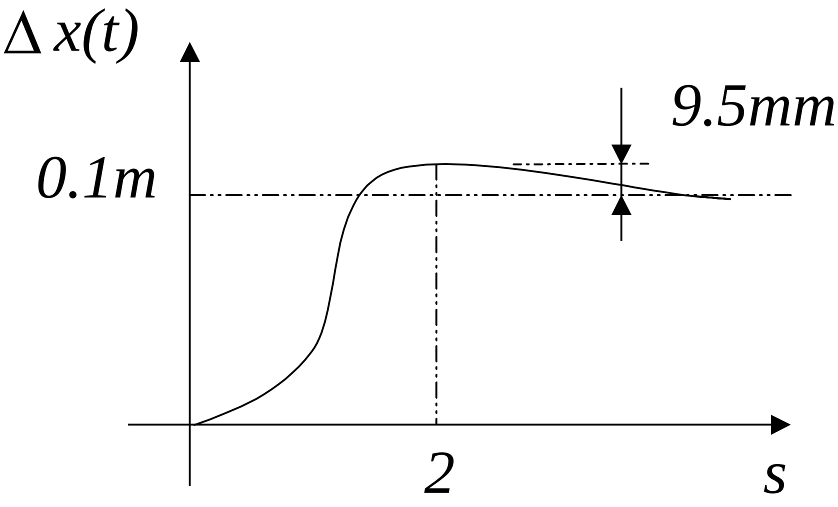
<u>$y_9(t)=1-\cos(t)$</u>

```
>>g9=tf(1,[1 0 1])
>>step(g9)
```

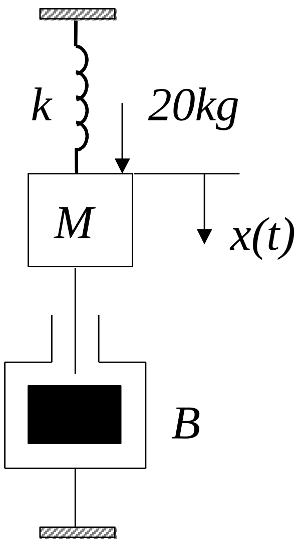
Ejercicio 6.9

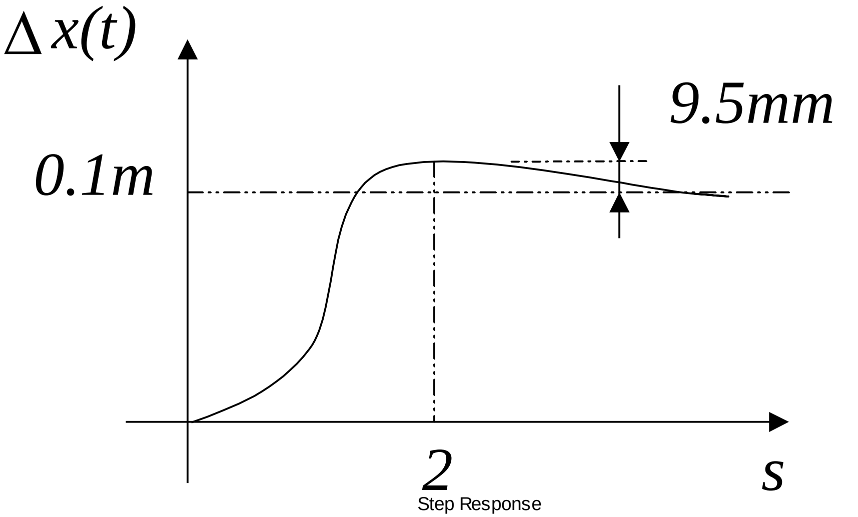
El sistema de la figura responde ante una aplicación brusca de una fuerza de 20kg apartándose de su posición de equilibrio como se indica a continuación. Determinar M, B y k.

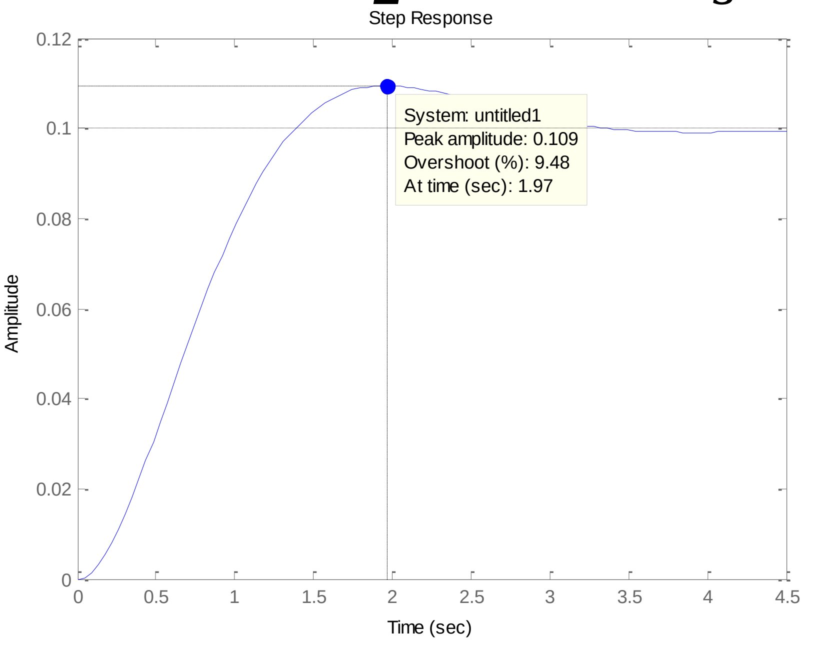




Ejercicio 6.9







$$k = 2000 \left[\frac{N}{m}\right]$$

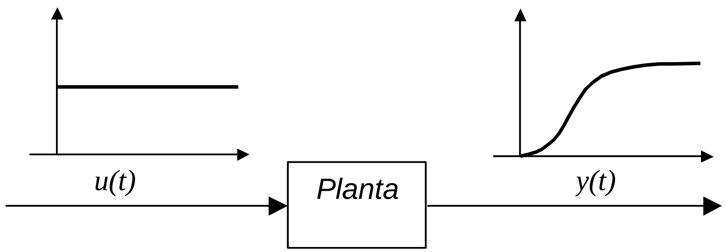
$\omega_d = \frac{\pi}{2}$ $\quad M_p = 9.5\% \rightarrow \theta = 53°$

$$\omega_n = 2 \qquad \xi = 0.6$$

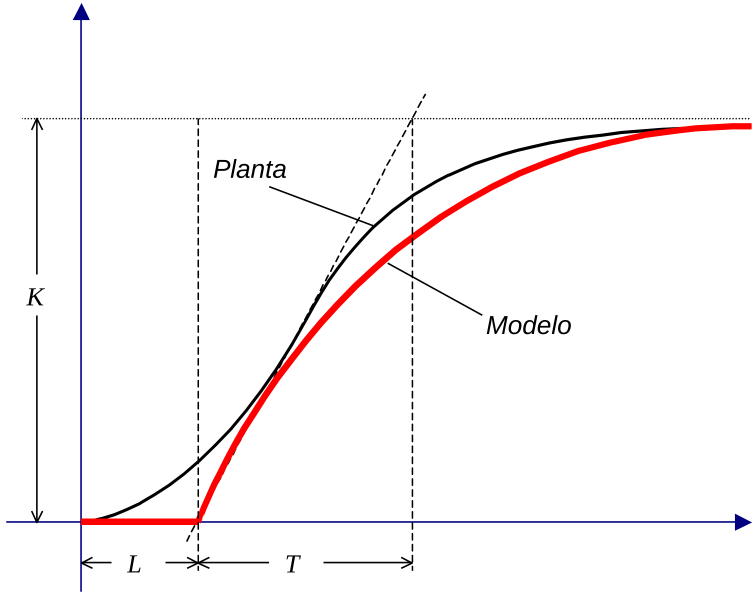
$$M = 500kg \quad B = 1200 \left[\frac{Ns}{m}\right]$$

Plantas Ziegler-Nichols

- Modelo & experimentación



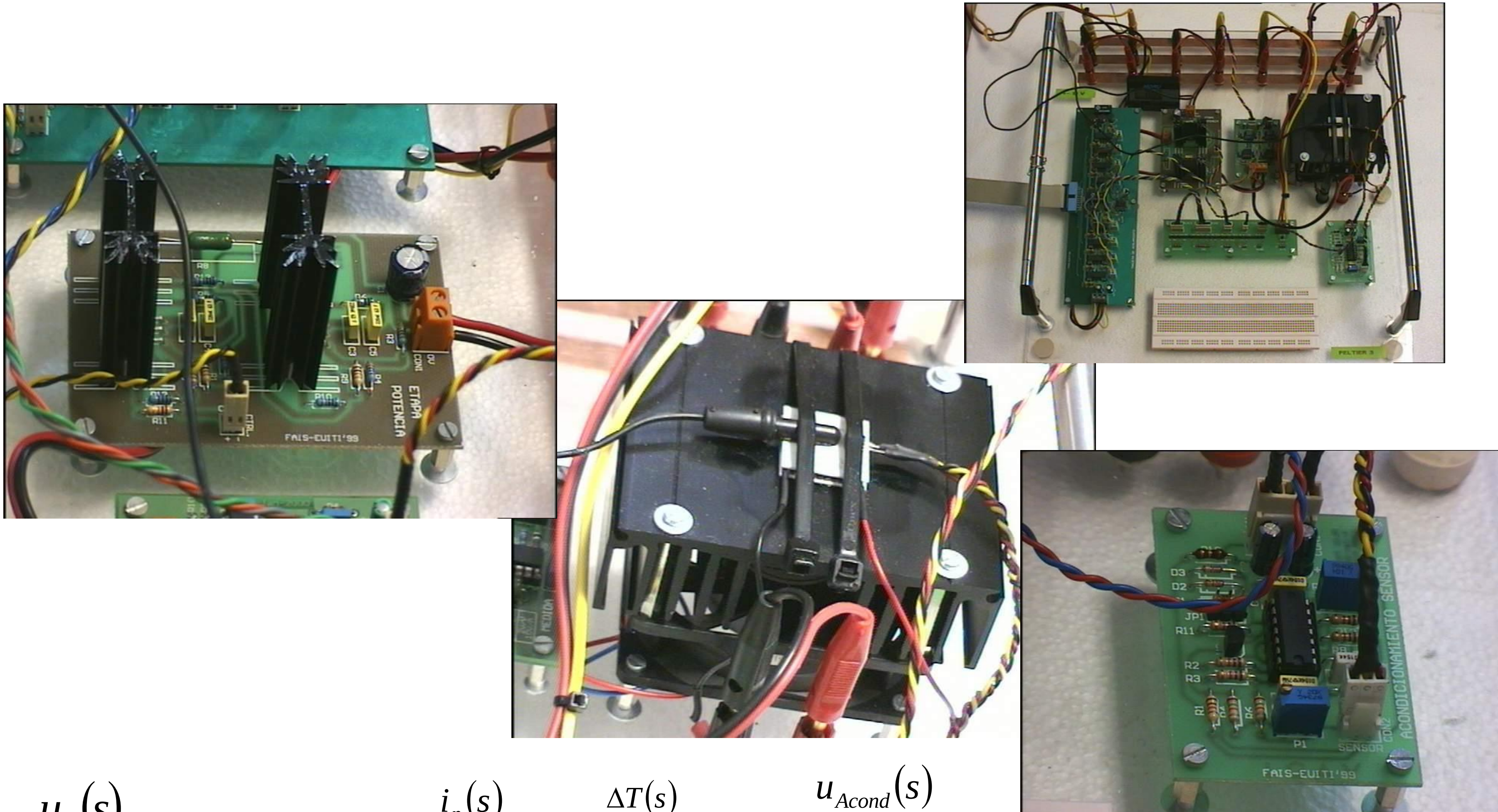
$$G_p(s) \approx e^{-sT_d}\,\frac{k}{1+sT}$$

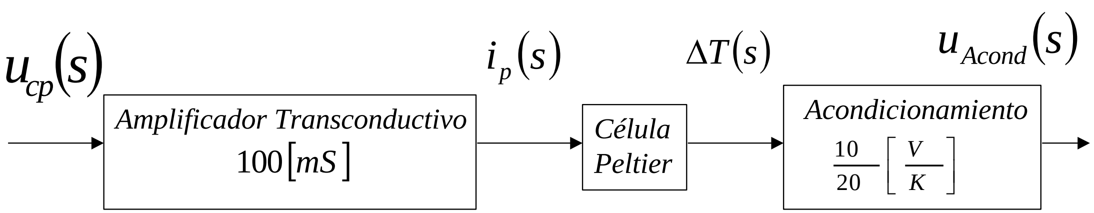


- Aproximación de Pade:

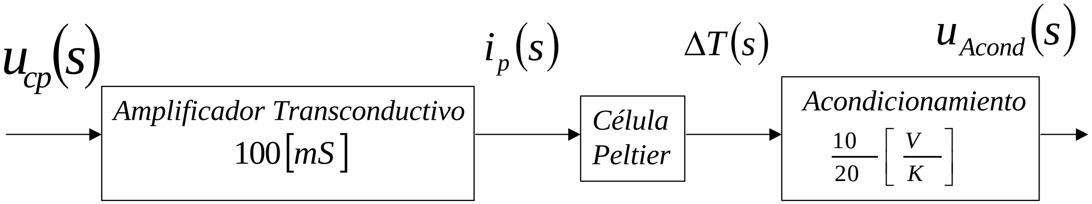
$$e^{-sT_d} = \frac{1-T_d\,\frac{s}{2}}{1+T_d\,\frac{s}{2}}$$

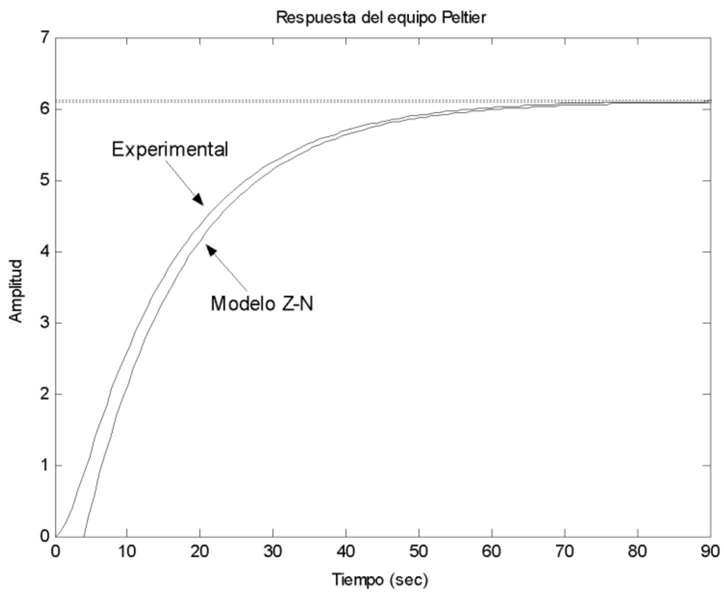
El equipo Peltier

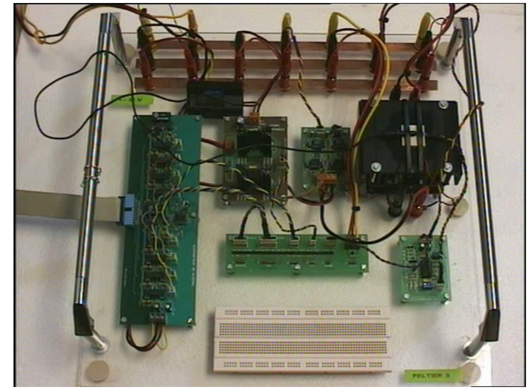




Modelo Ziegler-Nichols







$$k = \frac{6.12}{5} = 1.22$$

T_d=4s

$$3T = 45 - 4 = 41s \rightarrow T = \frac{41}{3} = 13.66s$$

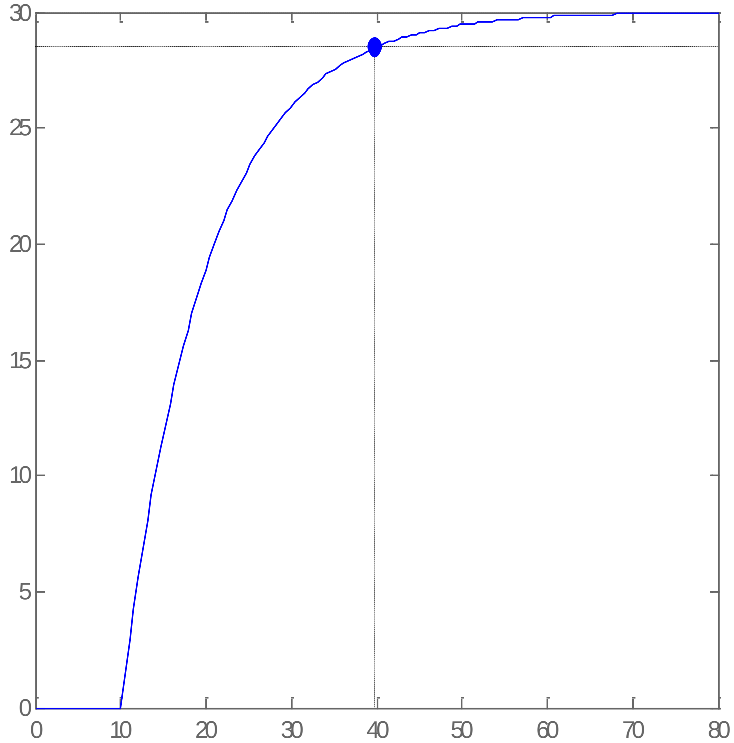
Simplificado: $$G_p(s) \approx \frac{1.22/(4 \cdot 13.66)}{(s+0.25)(s+0.073)} = \frac{0.09}{(s+0.25)(s+0.073)} \approx \frac{0.045}{(s+0.07)(s+0.525)}$$

Pade: $$G_p(s) = e^{-4s}\frac{1.22}{1+13.66s} \cong \frac{1-2s}{1+2s}\frac{1.22}{1+13.66s} = -\frac{s-0.5}{s+0.5}\frac{0.09}{s+0.073}$$

Ejercicio

Dibujar la respuesta al escalón del sistema de:

$$G(s) = 3\frac{e^{-10s}}{s+0.1}$$



g=tf(3,[1 0.1],'InputDelay',10);
step(g)